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Charge Moving with the Speed of Light in Einstein-Maxwell Theory

W. B. BONNOR

Queen Elizabeth College, London

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Abstract

I give solutions of the Einstein-Maxwell equations describing charge moving with the speed of light, c. The motion generates plane-fronted electromagnetic and gravitational waves. Charges moving parallel to each other with speed c do not interact; nor do they interact with parallel light beams.

1. *Introduction*

In a previous paper (Bonnor, 1969b), I gave solutions of Maxwell's equations describing charge moving with the speed of light, c (called a *null current*). I now give the corresponding solutions in the Einstein-Maxwell theory (EMT). This is worth doing, because whereas Maxwell's theory (MT) allows one to place charge where one likes, EMT does not. For instance, MT admits a static solution for two isolated charges at rest, but EMT does not. Put another way, EMT entails the equations of motion of matter (charged and uncharged), whereas MT does not (even for charged matter).

It turns out that the previously given solutions of MT have precise analogues in EMT. This means that they are possible mechanically, as well as electromagnetically. For example, MT allows a dipole moving with speed c; an analogous solution exists in EMT, and has no non-electromagnetic stresses. It seems at first sight surprising that positive and negative charge can co-exist in steady motion. The explanation is that for this motion at speed c the ponderomotive force vanishes, so no non-electromagnetic stresses are necessary to enforce the motion. In fact the ponderomotive force vanishes in the whole class of solutions given previously (Bonnor, 1969b), which is why they have close analogues in EMT.

This lack of interaction between objects moving in straight lines† with speed c was noted previously in the case of pulses and beams of light

† I shall sometimes use this graphic, though loose, description. Strictly, the elements of the objects move on null geodesics. However, if one neglects the gravitational field and considers only the background Minkowski space-time, the projections of the world-lines of the elements on to a space-like 3-section are straight lines.

(Bonnor, 1969a). It is an easy matter to include such pulses and beams in this work, which allows me to show that they do not interact with parallel null currents.

We shall, therefore, be interested in solutions of Einstein's equations with sources of two types:

(i) an electromagnetic field generated by a null current;
(ii) a directed flow of radiation (also called a *null fluid*).

The gravitational field equations therefore are

$$R^{ik} - \tfrac{1}{2}g^{ik}R = -8\pi E^{ik} - 8\pi L^{ik} \tag{1.1}$$

where

$$E^{ik} = -F^{ia}F^{k}{}_{a} + \tfrac{1}{4}g^{ik}F^{ab}F_{ab} \tag{1.2}$$

is the energy tensor for an electromagnetic field F_{ik}, and where

$$L^{ik} = \rho s^i s^k \tag{1.3}$$

is the energy tensor for the null fluid of energy density ρ and null velocity s^i. Maxwell's equations are

$$F_{ik} = B_{i,k} - B_{k,i} \tag{1.4}$$

$$4\pi J^i = F^{ik}{}_{;k} \tag{1.5}$$

B_i being the vector potential and J^i the null four-current.

Solutions in which only sources (ii) are present have been studied previously (Bonnor, 1969a). I must emphasise that the null fluid is only a phenomenological representation of a pulse of light: the Maxwell equations for the pulse are *not* satisfied. I adopt the standpoint of geometrical optics and represent the pulse by the energy tensor (1.3). This gives a representation of the gravitational field of the pulse, whilst neglecting its electromagnetic field. For something like a laser pulse this is probably not a bad model. Use of the null fluid concept has often been made in the past (e.g. Tolman, 1934; Vaidya, 1953).

The fields created by the null currents are plane-fronted electromagnetic and gravitational waves, as is shown in Section 2. In Section 3, I describe the superposition property of the fields, which seems to me the most interesting feature of the work. In Section 4 a solution for a pulse of null fluid is given, and in Section 5 some solutions for charges moving with speed c in a straight line. The latter are members of a class previously discovered by Peres (1960) and Takeno (1961), and they are also closely related to the solutions of Wyman & Trollope (1965); but, as far as I am aware, my interpretation of them is new.

When speaking of continua it is sufficient to refer to null fluid, and null current, as defined above. However, it is useful to have words describing bodies made out of these continua, and for these I propose *nullicon* and *charged nullicon*, respectively. A nullicon or a charged nullicon, then, is a body each of whose points has a null velocity vector. A nullicon is a photon stripped of its electromagnetic properties. A charged nullicon may have $\rho = 0$ (zero 'bare mass'), but it must nevertheless have a positive gravitational mass because of its electromagnetic energy (Section 5).

2. *Plane-Fronted Electromagnetic and Gravitational Waves*

We use the metric (Ehlers & Kundt, 1962)

$$ds^2 = -dx^2 - dy^2 + 2\,du\,dv + 2A\,du^2 \tag{2.1}$$

where $-\infty < x,\ y,\ u,\ v < \infty$. The function $A(x,y,u)$ is to be of class C^1, piecewise C^3. The coordinates will be numbered

$$x^1 \equiv x, \qquad x^2 \equiv y, \qquad x^3 \equiv v, \qquad x^4 \equiv u \tag{2.2}$$

so that x^1 and x^2 are space-like, and x^3 null and x^4 are time-like, if $A > 0$. The metric (2.1) has determinant

$$g = -1 \tag{2.3}$$

and the only non-zero component of its Ricci tensor is

$$R^{33} = -(A_{11} + A_{22}) \tag{2.4}$$

The curvature scalar satisfies

$$R = 0 \tag{2.5}$$

because of (2.1) and (2.4).

The transformation

$$\sqrt{(2)}\,u = t - z, \qquad \sqrt{(2)}\,v = t + z \tag{2.6}$$

takes (2.1) into

$$ds^2 = -dx^2 - dy^2 - dz^2(1 - A) - 2A\,dz\,dt + (1 + A)\,dt^2 \tag{2.7}$$

When the metric is used in this form I shall write

$$\bar{x}^1 \equiv x, \qquad \bar{x}^2 \equiv y, \qquad \bar{x}^3 \equiv z, \qquad \bar{x}^4 \equiv t \tag{2.8}$$

and tensors referred to these components will have a bar over them, e.g. $\bar{R}_{ik}$.

The electromagnetic field will be generated by the vector potential

$$B^i = (0, 0, \phi, 0) \qquad \text{or} \qquad B_i = (0, 0, 0, \phi) \tag{2.9}$$

where $\phi(x,y,u)$ is of class C^1, piecewise C^3. This is sufficient to ensure continuity of the electromagnetic field, and so to avoid surface charges and surface currents. The field F_{ik}, given by (1.4), has non-zero components

$$-F_{14} = F_{41} = \phi_1, \qquad -F_{24} = F_{42} = \phi_2 \tag{2.10}$$

where a subscript 1 or 2 after ϕ means $\partial/\partial x$ or $\partial/\partial y$. The current J^i, given by (1.5), has components

$$J^i = (4\pi)^{-1}(0, 0, -\nabla^2\phi, 0) \stackrel{\text{def}}{=} (0, 0, j, 0) \tag{2.11}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.12}$$

J^i is a null vector and provided that, at each world-point, charge of only one sign is present, *it represents charge moving in a straight line with the speed of light*, as shown previously (Bonnor, 1969b). The field (2.10) is null, i.e.

$$F^{ik} F_{ik} = 0, \qquad \eta^{iklm} F_{ik} F_{lm} = 0 \tag{2.13}$$

and the electromagnetic energy tensor E^{ik} has only one non-zero component, namely

$$E^{33} = \phi_1{}^2 + \phi_2{}^2 \tag{2.14}$$

We can add an energy tensor L^{ik} given by (1.3) with

$$s^i = \sqrt{(2)}\, \delta_3{}^i \tag{2.15}$$

L^{ik} has as only non-zero component

$$L^{33} = 2\rho \tag{2.16}$$

$\rho(x,y,u)$ being the energy density of the null fluid; ρ is assumed to be piecewise of class C^0 and non-negative. The factor 2 is required because, for physical reasons, the energy density is to be identified as follows

$$-\bar{L}_3{}^3 = -\bar{L}_3{}^4 = \bar{L}_4{}^3 = \bar{L}_4{}^4 = \rho \tag{2.17}$$

in the coordinates of (2.7) (Tolman, 1934; Bonnor, 1969a), and $L^{33} = 2\bar{L}_4{}^4$.

From (2.4), (2.14) and (2.16) we have as the only field equation

$$A_{11} + A_{22} = 8\pi(\phi_1{}^2 + \phi_2{}^2 + 2\rho) \tag{2.18}$$

In space empty of charges and null fluid we have, as well,

$$\nabla^2 \phi = 0 \tag{2.19}$$

$$\rho = 0 \tag{2.20}$$

If (2.19) is not satisfied there is a null current given by (2.11). To sum up, we have shown that *a solution of the Einstein-Maxwell equations is given by the metric* (2.1) *together with equations* (2.10), (2.16) *and* (2.18); *and in empty space* (2.19) *and* (2.20) *must be satisfied too*. The solution gives the field of straight null currents given by (2.11), and null uncharged fluid of density ρ and velocity (2.15); it represents combined electromagnetic and gravitational plane-fronted waves, the fronts being the null hypersurfaces $u = \text{const}$. The waves are like plane waves, but the fields may depend on x, y over each wave-front (Kundt, 1961).

We may satisfy (2.18) by writing

$$A = 4\pi\phi^2 + A^* + A^{**} \tag{2.21}$$

where

$$A^*_{11} + A^*_{22} = 32\pi^2 \phi j \tag{2.22}$$

$$A^{**}_{11} + A^{**}_{22} = 16\pi\rho \tag{2.23}$$

and j is given by (2.11). The problem now becomes determinate if distributions of null current j and null fluid of density ρ are given. ϕ is uniquely determined by (2.11) and suitable conditions at $x, y = \infty$; A^*, A^{**} are then determined by (2.22) and (2.23), and are unique if certain conditions on them at $x, y = \infty$ are prescribed. Hence, from (2.21), A is uniquely determined given j and ρ.

I shall actually relax the conditions imposed at $x, y = \infty$ to allow solutions corresponding to monopole charge and positive mass moving with speed c. The latter require ϕ and $A \sim \log(x^2 + y^2)$, so that one may add to both ϕ and A an arbitrary function of u. This destroys the uniqueness of ϕ and A, given j and ρ. Physically, this is of no importance, because the electromagnetic field F_{ik} is independent of such additions to ϕ and the addition to A can be removed by a coordinate transformation of the type:

$$x = x, \qquad y = y, \qquad v = \bar{v} + \chi(u), \qquad u = u \tag{2.24}$$

Examples of solutions are given in Sections 4 and 5.

3. *Superposition of Solutions*

An interesting feature of the solutions described in Section 2 is that they may be superposed without the introduction of singularities in the metric. To see this, consider two solutions $(\overset{(1)}{A}, \overset{(1)}{\phi}, \overset{(1)}{\rho})$, $(\overset{(2)}{A}, \overset{(2)}{\phi}, \overset{(2)}{\rho})$ satisfying (2.18), and in empty space (2.19) and (2.20) also. Then the superposed solution is (A, ϕ, ρ) given by

$$\phi = \overset{(1)}{\phi} + \overset{(2)}{\phi}, \qquad \rho = \overset{(1)}{\rho} + \overset{(2)}{\rho}, \qquad j = \overset{(1)}{j} + \overset{(2)}{j} \tag{3.1}$$

$$A = 4\pi\phi^2 + A^* + A^{**} \tag{3.2}$$

where A^* and A^{**} satisfy (2.22) and (2.23). Assuming now (in accordance with the hypotheses of Section 2) that j and ρ are piecewise of class C^0, and that they are zero outside a bounded region in the x, y plane, it follows that (2.22) and (2.23) admit solutions A^*, A^{**} which are of class C^1 in every closed region (the proof follows by minor modifications of textbook results, e.g. Kellogg, 1953, p. 150). Hence A given by (3.1) is of class C^1, and the complete solution is regular. Also, since the right-hand side of (2.18) is evidently non-negative, sources of negative energy are not present. *Hence the superposition is non-singular, and no stresses are required to hold the sources in position*; it is also physically reasonable, at least in the sense that negative energy is not included.

That there is no electromagnetic interaction of the charges is confirmed by the vanishing of the mechanical force: we have, using (2.10) and (2.11)

$$F_{ik}J^i = 0 \tag{3.3}$$

whatever ϕ may be.

The foregoing shows that *fields created by* ρ *and* J^i *show almost no interaction.* To be explicit,

(i) the null fluid of density ρ does not interact with itself or with the electromagnetic field;

(ii) the electromagnetic field shows no self-interaction (i.e. the charges do not influence each others' motion), but does produce extra terms in the gravitational potential.

It is to be emphasised that these conclusions apply only if the charges move in a straight line with speed c parallel to each other and to the null fluid, and all in the same sense.

4. *Field of a Pulse of Null Fluid* (*Nullicon*)

In this section I put $\phi = 0$ and give the field of a pulse of null fluid, since this will be needed in Section 5. The solution for this is (cf. Bonnor, 1969a) given by substituting the following function A in (2.1)

$$A_{\text{ext}} = \psi\left(2\log\frac{r}{a} + 1\right), \qquad \rho = 0, \qquad r \geqslant a \tag{4.1}$$

$$A_{\text{int}} = \frac{\psi r^2}{a^2}, \qquad \rho = \frac{\psi}{4\pi a^2}, \qquad r \leqslant a \tag{4.2}$$

where $\psi(u)$ is a C^3 pulse-function of the type described in Section 5, e.g. by (5.4). The occurrence of the logarithmic term does not render the solution unphysical, for reasons to be given in Section 5. The term in $\log r/a$ represents the effect of the gravitational mass of the nullicon.

The non-uniqueness, referred to in Section 2, can be seen in this solution: we could instead have taken

$$A_{\text{ext}} = 2\psi\log\frac{r}{a}, \qquad r \geqslant a$$

$$A_{\text{int}} = \psi\left(\frac{r^2}{a^2} - 1\right), \quad r \leqslant a$$

This solution can be transformed into (4.1)–(4.2) by choosing a new coordinate $\bar{v}$ such that

$$v = \bar{v} + \tfrac{1}{2}\int \psi\, du$$

5. *Charges Moving with the Speed of Light* (*Charged Nullicons*)

In this section I put $\rho = 0$.

First I shall give the solution for charge† of one sign moving parallel to

† Strictly, all that is needed is that on each null hypersurface $u = u_0$ (const.) the sign of J^i is preserved; for different values of u_0 the sign can differ. This is referred to again later in this section.

the z-axis with the speed of light. Take

$$\phi = \psi(u)\left(2\log\frac{r}{a}+1\right), \qquad A = 4\pi\psi^2\left\{4\left(\log\frac{r}{a}\right)^2+2\left(\log\frac{r}{a}\right)+\tfrac{1}{2}\right\}, \qquad r \geqslant a \tag{5.1}$$

$$\phi = \frac{\psi(u)\,r^2}{a^2}, \qquad A = \frac{2\pi\psi^2 r^4}{a^4}, \qquad r \leqslant a \tag{5.2}$$

where $r = +(x^2+y^2)^{1/2}$. This solution satisfies (2.18) and (2.20) for $r \geqslant 0$, and (2.19) for $r > a$; it is of class C^1, piecewise C^3, in any closed interval of r, provided $\psi(u)$ is of class C^3. It therefore is a solution of the Einstein-Maxwell equations. Its source is the null current

$$\begin{aligned} J^i &= 0, & r > a \\ J^i &= -\frac{\psi}{\pi a^2}\delta_3{}^i, & r < a \end{aligned} \tag{5.3}$$

ϕ and A tend to infinity with r, but the electromagnetic field, given by (2.10), tends to zero. At every world-point P the metric (2.1) [with A given by (5.1)] can be transformed to natural (freely falling) coordinates, in which the g_{ik} have Minkowski values at P and the derivatives of g_{ik} vanish at P. The geodesics and null geodesics are therefore perfectly well behaved, and the solution is admissible in spite of the logarithmic terms.

If ψ is constant, the solution (5.1)–(5.3) refers to a steady stream of charge moving with the speed of light; the electric field is that of an infinite line charge, and the magnetic field that of an infinite straight current.

One can construct a model of a charged nullicon by using (5.1) and (5.2) to form F_{ik} and g_{ik}, and by choosing for $\psi(u)$ some suitable C^3 pulse-function, e.g.

$$\begin{aligned} \psi(u) &= (b^2-u^2)^4, & |u| \leqslant b > 0 \\ \psi(u) &= 0, & |u| \geqslant b \end{aligned} \tag{5.4}$$

This particle contains nothing but charge of negative sign, yet unlike a static charged particle, it needs no non-electromagnetic stresses to hold it together. This is expected because of (3.3).

For a pulse such as (5.4) an observer O at $P(x_0, y_0, z_0)$ experiences the electromagnetic and gravitational forces only whilst the pulse is passing P in its journey along Oz. The sharper the pulse the shorter the time during which O experiences the fields, but (if P is outside the pulse) O finds during this short time the electric field of an infinite line charge, and the magnetic field of an infinite straight current. This behaviour was analysed previously (Bonnor, 1969b).

We can use (5.1) and (5.2) to represent two charged nullicons moving along Oz in the positive sense. Take ψ to consist of two non-overlapping pulses, e.g.

$$\begin{aligned} \psi &= (b^2-u^2)^4, & -b \leqslant u \leqslant b \\ \psi &= \{f^2-(u-d)^2\}^4, & d-f \leqslant u \leqslant d+f \\ \psi &= 0 \text{ for other values of } u & \end{aligned} \tag{5.5}$$

where b, d, f are positive, and

$$d - f > b$$

Inspecting A given by (5.1), (5.2) and (5.5), we see that *the two charged nullicons do not interact at all*; they do not even produce an extra term in A (see Section 3). The solution would still apply if the two parts of ψ had opposite signs, thus giving oppositely charged nullicons.

Although the solution generated by (5.1) and (5.2) is globally regular, it is unphysical in that its electromagnetic energy

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E^{33}\sqrt{(-g)}\,dx\,dy\,du$$

[E^{33} is given by (2.14)] diverges, owing to the presence of the $\log r$ in (5.1). This infinite energy is reflected in the dominant term in A in (5.1), namely $(\log r/a)^2$; for large r, this is greater than the dominant term $\log r/a$ in (4.1), which refers to a nullicon of finite energy content. (The gravitational field is, however, well behaved in spite of the infinite electromagnetic energy.)

Globular regular solutions of finite electromagnetic energy do exist however: for instance the following, which represents a nullicon with a dipole moment but zero total charge

$$\phi = \frac{\psi(u)\cos\theta}{r}, \qquad A = \pi\psi^2\left(\frac{2\cos^2\theta}{7r^2} + \frac{13}{7r^2} + \frac{48}{5a^2}\log\frac{r}{a}\right), \qquad r \geqslant a \quad (5.6)$$

$$\phi = \frac{\psi\cos\theta}{a}\left[-3\left(\frac{r}{a}\right)^3 + 4\left(\frac{r}{a}\right)^2\right], \qquad r \leqslant a$$

$$A = \frac{\pi\psi^2}{a^2}\left\{\cos^2\theta\left[18\left(\frac{r}{a}\right)^6 - \frac{320}{7}\left(\frac{r}{a}\right)^5 + 32\left(\frac{r}{a}\right)^4 - 4\left(\frac{r}{a}\right)^2\right]\right. \quad (5.7)$$

$$\left. + \left[\left(\frac{r}{a}\right)^6 - \frac{704}{175}\left(\frac{r}{a}\right)^5 + 4\left(\frac{r}{a}\right)^4 + 2\left(\frac{r}{a}\right)^2 - \frac{28}{25}\right]\right\}, \qquad r \leqslant a$$

ψ being a pulse function. It will be noticed that although ρ is zero in this solution, A has a $\log r/a$ term which, according to Section 4, represents a gravitational mass. This is to be expected because the electromagnetic energy exerts a gravitational field.

6. *Conclusion*

The main conclusions of this work are as follows.

(1) Everywhere—regular solutions of Einstein-Maxwell theory exist representing straight null currents (charge moving in straight lines with the speed of light), and these currents generate plane-fronted electromagnetic and gravitational waves.

(2) Two or more parallel straight null currents (of either sign) can coexist without non-electromagnetic stresses. This is suggested by the fact that the

ponderomotive force vanishes; it is confirmed by the existence of rigorous solutions of EMT.

(3) Straight null currents do not interact with parallel beams and pulses of light.

(4) It is known (Lichnerowicz, 1955) that the energy tensor of every null electromagnetic field can be put in the form (1.3). Since the electromagnetic fields in this paper are null, it follows that the E^{ik} in (1.1) can be put in form (1.3). Looked at from this point of view, the null currents studied in the paper each generate a null fluid.

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